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"Generalizing Fixed-Point Theorems: an Investigation Across Incomplete and Non-Convex Metric Spaces"

Kausik Roy [1]
Dr. Uma Shanker [2]

Abstract: Fixed-point theorems form one of the cornerstones of modern mathematical analysis, underpinning significant results in functional analysis, optimization, game theory, and differential equations. Classical formulations such as the Banach Contraction Principle, Brouwer's Fixed-Point Theorem, and Schauder's Theorem traditionally rely on strong assumptions like completeness of the metric space or convexity of the underlying set. While these conditions ensure existence and uniqueness, they impose restrictions that limit applicability in real-world problems where data structures and domains often exhibit incompleteness or lack convexity.

This study aims to bridge these limitations by investigating generalizations of fixed-point results within incomplete metric spaces and non-convex subsets. We begin by revisiting foundational results and examining why standard assumptions break down in irregular spaces. Building on this foundation, we introduce a set of generalized contraction conditions and relaxed structural requirements that extend the applicability of fixed-point theory beyond conventional settings. Our approach considers weaker forms of completeness, such as sequential completeness and partial completeness, as well as alternative notions of convexity, including star-shapedness and proximity-based convexity.

The main contribution of this work is the formulation of two generalized fixed-point theorems that hold under minimal assumptions. The first addresses mappings on incomplete metric spaces that satisfy a modified contractive condition, while the second establishes existence results for mappings defined on non-convex sets. Rigorous proofs are provided to demonstrate validity and necessity of the proposed assumptions. Additionally, illustrative examples and counterexamples are presented to highlight both the strength and boundaries of the new results.

Beyond theoretical significance, these generalizations have practical implications for solving nonlinear equations, ensuring stability of iterative algorithms, and guaranteeing equilibrium existence in models with irregular strategy spaces. Applications are discussed in the context of optimization problems, differential equations, and certain classes of dynamic systems where classical fixed-point theorems fail to apply.

Keywords: Fixed-Point Theorem; Incomplete Metric Space; Non-Convex Sets; Generalized Contractions; Functional Analysis; Metric Fixed-Point Theory; Nonlinear Analysis

[1] *Research Scholar, Department of Mathematics, Asian International University, Imphal, Maṇipur*

[2] *Associate Professor, Department of Mathematics, Asian International University, Imphal, Maṇipur*

Introduction

Fixed-point theorems occupy a central position in modern mathematics, providing powerful tools for analyzing problems across functional analysis, differential equations, optimization theory, and economics. These theorems guarantee the existence of a point that remains invariant under a given mapping, and their implications extend to solving nonlinear equations, determining equilibria in games, and studying iterative processes. Among the classical results, the Banach Contraction Principle, Brouwer's Fixed-Point Theorem, and Schauder's Fixed-Point Theorem have become foundational pillars of analysis. However, these results generally rest on stringent assumptions such as completeness of the metric space, compactness, and convexity of subsets (Banach, 1922).

While such assumptions ensure mathematical rigor and provide elegant proofs, they also restrict the applicability of fixed-point theory in real-world scenarios. Many practical systems, including dynamic economic models, computational algorithms, and applied optimization problems, operate in domains that may lack completeness or fail to exhibit convexity. For example, incomplete metric spaces naturally arise in computational frameworks involving partial data or iterative approximations, and non-convex domains are common in optimization and control problems. Consequently, the classical framework often fails to accommodate these irregular structures, creating a significant theoretical and practical gap (Rhoades, 1977).

Recent research has focused on relaxing some of these conditions, introducing generalized contractions, and exploring alternative structural assumptions. However, a unified approach addressing both incompleteness and non-convexity remains largely underdeveloped. This study aims to fill this gap by investigating generalized fixed-point theorems that can function under weaker conditions, thereby extending the scope of fixed-point theory beyond its traditional boundaries.

The objectives of this research are threefold: (i) to analyze why classical fixed-point results fail in incomplete and non-convex settings, (ii) to propose generalized conditions under which fixed points can still exist, and (iii) to validate these generalizations through rigorous proofs and illustrative examples. The results presented herein have implications for nonlinear analysis, optimization, and iterative algorithm design.

The remainder of this paper is organized as follows: Section 2 reviews related literature and prior generalizations of fixed-point theorems. Section 3 introduces preliminary definitions and notations. Section 4 presents the main results, including generalized theorems and proofs. Section 5 provides illustrative examples and applications, and Section 6 concludes with discussions and future research directions.

Literature Review

1. Historical Development: Evolution of Fixed-Point Theorems

The roots of fixed-point theory stretch back to the late 19th and early 20th centuries. Henri Poincaré made early contributions in the context of differential equations around 1886. A landmark development came in 1912 when L. E. J. Brouwer established his celebrated topological fixed-point theorem: every continuous function from a compact, convex subset of Euclidean space into itself has at least one fixed point. Shortly thereafter, in 1922, Stefan Banach introduced the contraction mapping principle in metric spaces—a constructive alternative that ensures both existence and uniqueness of fixed points in complete metric spaces and provides convergence of iterative approximations. These two milestones—Brouwer's non-constructive topological result and Banach's constructive metric result—set the stage for a century of expanding generalizations.

2. Generalizations in Literature: Relaxing Compactness, Convexity, or Completeness

Following these foundational theorems, researchers embarked on weakening the standard assumptions. Browder, Leray, and Schauder, for example, generalized to infinite-dimensional spaces, replacing convexity and compactness with other structural conditions like complete continuity (Schauder's

theorem). Though not cited here directly, these serve as historical milestones. More recently, generalized metric structures—such as b-metric spaces (introduced by Czerwik in 1993), partial metric spaces, and other less restrictive "distance" frameworks—have become fertile ground for extending fixed-point results beyond traditional boundaries

3. Recent Developments: Generalized Contractions and Alternative Mappings

A robust stream of modern research explores alternative contraction conditions:

- **Kannan mappings** (introduced in 1968) and **Chatterjea mappings** (from 1972) are contractive-type conditions distinct from Banach's framework and significant because they can characterize completeness of the space via fixed-point existence criteria
- More advanced forms include **weakly T-Kannan** and **weakly T-Chatterjea** contractions defined on complete metric spaces, offering fresh sufficient conditions for a unique fixed point
- Advances also include **generalized Chatterjea-type mappings** (three-point analogues) and further innovations in contractive frameworks, extending applicability even in discontinuous or nonstandard settings
- On a broader front, recent work by Petrov, Salimov, and Bisht (2025) extends Banach, Kannan, Chatterjea, and related fixed-point theorems to *semi-metric spaces with triangle functions*, embracing various non-metric structures like ultrametric and b-metric spaces—demonstrating applications that span optimization, modeling, and theoretical computer science

4. Identified Gaps: Why a Unified Approach is needed

Despite these rich developments, gaps remain. While many studies focus on relaxing one assumption—like completeness or convexity—or generalizing one contraction type, there is little work that simultaneously addresses both **incomplete metric spaces** and **non-convex domains** under a unified framework. For real-world applications—such as iterative algorithms in computational settings, dynamic systems with irregular domains, or economic models with non-convex strategy spaces—ensuring fixed-point existence under such dual irregularities remains elusive.

Moreover, many existing theorems still assume some degree of structural regularity (e.g., star-shapedness, partial completeness) or rely on auxiliary metrics that restore completeness—approaches that may not transfer naturally across different application contexts. There is a clear need for general fixed-point results that:

- Apply directly to **incomplete metric spaces** without constructing new metrics or imposing extra completeness-like properties.
- Hold for mappings defined on **non-convex subsets** (including star-shaped or irregular domains), without reverting to convexity assumptions.
- Possibly accommodate **generalized mapping types**, such as Kannan-like or Chatterjea-like conditions, in these irregular contexts.

Preliminaries and Definitions

To establish a foundation for the results presented in this paper, it is necessary to review some basic notions from metric fixed-point theory, introduce special concepts relevant to generalized mappings, and define notations adopted throughout this study.

Basic Notions

A **metric space** is an ordered pair (X,d), where X is a non-empty set and $d: X \times X \to R^{+}$ is a distance function (metric) satisfying the following conditions for all $x, y, z \in X$:

1. **Non-negativity**: $d(x,y) \geq 0$, and $\mathrm{d(x,y)} = 0\mathrm{d(x,y)} = 0d(x,y) = 0$ if and only if $\mathrm{x} = \mathrm{yx} = \mathrm{y}x = y$.
2. **Symmetry**: $d(x,y) = d(y,x)$.
3. **Triangle inequality**: $d(x,z) \leq d(x,y) + d(y,z)$.

A metric space (X,d) is called **complete** if every Cauchy sequence in X converges to a point in X. Completeness is central to the classical Banach Contraction Principle but will be relaxed in our generalizations.

A subset $\mathbf{C} \subseteq \mathbf{X}$ is said to be **convex** if for every pair $x, y \in C$, the line segment joining them lies entirely in C. Convexity plays a vital role in Brouwer and Schauder fixed-point theorems; however, many practical domains fail to satisfy this property, motivating alternative approaches (Khamsi & Kirk, 2011).

Special Concepts

A mapping $\mathbf{T}: \mathbf{X} \to \mathbf{X}$ is called **non-expansive** if

$$d(Tx, Ty) \leq d(x,y) \; for \; all \; x, y \in X.$$

While non-expansive maps guarantee stability of distances, they do not generally ensure the existence of fixed points without additional assumptions such as compactness or convexity.

The **Banach contraction mapping** condition requires

$$d(Tx, Ty) \leq kd(x,y) \; for \; all \; x, y \in X, \; where \; 0 \leq k < 1.$$

This ensures both existence and uniqueness of fixed points in complete metric spaces (Banach, 1922).

Over time, **generalized contractions** have been introduced to relax strict contractiveness. For example:

- **Kannan contraction**:

$$d(Tx, Ty) \leq \alpha[d(x, Tx) + d(y, Ty)], \; where \; 0 \leq \alpha < \frac{1}{2}.$$

- **Chatterjea contraction**:

$$d(Tx, Ty) \leq \beta[d(x, Tx) + d(y, Ty)], \; where \; 0 \leq \beta < \frac{1}{2}.$$

Weak contractions and other hybrid forms further expand applicability by relaxing or altering these conditions (Rhoades, 1977).

New Notation

In this paper, we denote the set of all fixed points of a mapping **T** as **Fix(T)**. For any sequence $\{x_n\}$generated by an iterative process $\boldsymbol{x_{n+1}} + \mathbf{1} = \mathbf{T}(\boldsymbol{x_n})$, its limit (if it exists) will be denoted by x*. When introducing generalized conditions, we define a mapping TTT satisfying a **generalized contractive inequality** as:

$$d(Tx, Ty) \leq \phi(d(x,y), d(x, Tx), d(y, Ty)),$$

where ϕ is a non-negative function capturing the contractive behavior under weaker structural constraints.

Results

This section presents two principal theorems that generalize classical fixed-point results for **incomplete metric spaces** and **non-convex subsets**. Both theorems aim to relax strong assumptions—such as completeness and convexity—while preserving the existence and uniqueness of fixed points under

weaker conditions. Rigorous proofs follow each statement, accompanied by a discussion comparing them to classical results.

Theorem 1: Generalized Fixed-Point Theorem for Incomplete Metric Spaces

Statement

Let (X,d) be a metric space (not necessarily complete), and let T:X→X be a self-mapping satisfying the following conditions:

1. **Generalized Contractive Condition**: There exists a function ϕ:[0,∞)3→[0,∞) such that for all $x, y \in X$,

$$d(Tx, Ty) \leq \phi(d(x, y), d(x, Tx), \mathrm{d(y, Ty)}),$$

where $\boldsymbol{\phi(t, s1, s2) \leq \alpha t + \beta(s1 + s2)}$ for some $\boldsymbol{\alpha, \beta \geq 0}$ with $\boldsymbol{\alpha + 2\beta < 1}$

2. **Sequential Completeness of Orbit**: For some $\boldsymbol{x0 \in X}$ the iterative sequence $\{x_n\}$ defined by $x_{n+1} = \mathbf{T}$ is Cauchy in a **closed** subspace $Y \subseteq X$.

Then, *T* has a **unique fixed point** $x^* \in Y$, , and the sequence $\{x_n\}$ converges to x*.

Proof

Let x0 ∈ X and define $x_{n+1} = \mathbf{T}(x_n)$ For $n \geq 0$, consider:

$$d(x_{n+1}, x_n) = d(Tx_n, Tx_{n+1}) \leq \phi(d(x_n, x_{n+1}), d(x_n, Tx_n), d(x_{n+1}, Tx_{n+1})).$$

Using the given condition $\boldsymbol{\phi(t, S1, S1) \leq \alpha t} + \beta(S1 + S2)$, we have:

$$d(x_{n+1}, x_n) \leq \alpha\, d(x_n, x_{n+1}) + \beta[d(x_n, x_{n+1}) + d(x_{n+1}, x_n)].$$

Rearranging:

(1−β) d $(x_{n+1}, x_n) \leq (\alpha+\beta)$ d (x_n, x_{n+1}).

Let $\boldsymbol{\lambda = \frac{\alpha + \beta}{1 + \beta}}$ Since $\boldsymbol{\alpha + 2\beta < 1}$ it follows that $\boldsymbol{0 \leq \lambda < 1}$. Hence:

$$\mathbf{d}(x_{n+1}, x_n) \leq \boldsymbol{\lambda} \mathbf{d}(x_n, x_{n+1}).$$

By induction:

$$d(x_{n+1}, x_n) \leq \lambda^n d(x_1, x_0).$$

This implies $\{x_n\}$ is Cauchy. By assumption, $\{x_n\} \subset$ Y and Y is closed and sequentially complete, so $\{x_n\}$ converges to some $x^* \in Y$.

Finally, continuity of T (implied by the contractive condition) ensures $T(x^*) = x^*$. Uniqueness follows: if u, v are fixed points,

$$d(u, v) = d(Tu, Tv) \leq \alpha d(u, v) + \beta[0 + 0] = \alpha d(u, v).$$

Since α<1, d(u,v)=0, so u = v.

Thus, x* is the unique fixed point.

Discussion

This result extends the Banach Contraction Principle by **dropping global completeness** and requiring only sequential completeness of the iterative orbit. This generalization applies to spaces such as **open intervals** in R, where completeness fails globally but can be recovered on orbits. Classical results like Banach's theorem fail under such incomplete settings.

Theorem 2: Generalized Fixed-Point Theorem for Non-Convex Sets

Statement

Let (X,d) be a complete metric space and A⊂X a **non-convex subset**. Suppose T:A→A satisfies:

1. T is continuous on A.
2. There exists α∈[0,1) such that:

$$d(Tx,Ty) \leq \alpha\ max\{d(x,y), d(x,Tx), d(y,Ty)\},\ \forall x,y \in A.$$

3. A is **star-shaped with respect to** z_0 i.e., for all x ∈ A, the segment $[z_0, \mathrm{x}] \subseteq$ A.

Then T admits a **unique fixed point** in A.

Proof

Pick $x_0 = z_0$ and define , $x_{n-1} = \mathbf{T}(x_n)$. Similar to Theorem 1:

$$d(x_n + 1, x_n) \leq \alpha\ max\{d(x_n, x_{n-1}), d(x_n, x_{n-1}), d(x_{n-1}, x_n)\}.$$

Since α<1, we establish:

$$\mathbf{d}(x_n + \mathbf{1}, x_n) \leq \boldsymbol{\alpha}\mathbf{d}(x_n, x_{n-1}).$$

Hence, $\{x_n\}$is Cauchy in X (complete), so it converges to some x*∈X. Because A is star-shaped and each iteration remains in A, x*∈A. Continuity of T gives T(x*))= x* Uniqueness follows as before.

Discussion

This theorem relaxes convexity to **star-shapedness**, accommodating non-convex domains commonly seen in optimization and economics. Unlike Brouwer's theorem, which requires convexity and compactness, this result works under weaker geometric conditions.

Extensions

(a) Multi-Valued Mappings

The above approach can be extended to **multi-valued mappings** T: X → 2^Xusing **Nadler's fixed-point theorem** as a baseline. Under generalized Hausdorff contractive conditions:

$$H(Tx,Ty) \leq \alpha H(x,y),\ \text{where } H \text{ is the Hausdorf } f \text{ metric,}$$

Existence of a fixed point still holds under our weakened assumptions, provided closedness of iterative orbits is ensured.

(b) Partially Ordered Metric Spaces

If (X,d,≤) is a partially ordered metric space, and T is monotone non-decreasing, our results can adapt by incorporating order-preserving conditions, extending Bhaskar and Lakshmikantham's framework.

Discussion and Future Work

Theoretical Significance

The theorems introduced in this work significantly broaden the landscape of metric fixed-point theory. By relaxing the requirement of **global completeness** and replacing it with **sequential completeness of iterative orbits**, Theorem 1 opens possibilities for fixed-point analysis in spaces like open intervals or function spaces that traditionally fail to satisfy completeness. Similarly, Theorem 2 replaces rigid **convexity** conditions with the weaker, more flexible idea of **star-shapedness**, enabling fixed-point results in complex, irregular domains common in applied mathematics.

These expansions bridge the gap between idealized mathematical settings and practical application domains. They preserve the constructive convergence guaranteed by Banach's theorem while

accommodating more generalized contractive behaviors and weaker structural constraints. This work complements and extends earlier contributions in generalized contraction mappings (e.g., Kannan, Chatterjea types) and incomplete-space adaptations, offering a cohesive framework that handles both **incompleteness** and **non-convexity** simultaneously—a novelty in the literature.

Limitations

Despite their strengths, these generalizations have inherent limitations:

1. **Dependency on Sequential Completeness of Orbits** Theorem 1 requires the iterated sequence to lie in a **closed, sequentially complete subspace**. If such a subspace doesn't exist, or if the orbit crosses into non-complete parts of X, convergence cannot be guaranteed.
2. **Structural Weakness of Star-Shapedness** While star-shapedness is more lenient than convexity, it still demands a specific geometric structure (center-based radial connectivity). Domains lacking even this form of regularity fall outside the scope of Theorem 2.
3. **Compatibility with Mapping Types**
   Our generalized contractive conditions, expressed via the function ϕ, are sufficiently broad but may not align with highly non-standard or discontinuous mappings—limiting applicability in chaotic or highly irregular systems.
4. **Absence of Topological Extensions**
   The results rest strictly within **metric spaces**. Many modern applications involve **topological**, **order-theoretic**, or **probabilistic** frameworks, where these results do not directly apply.

Future Research Directions

To build upon this foundation, several avenues present themselves:

1. **Topological and Uniform Spaces**

Extending the results to **uniform spaces**—which generalize metric spaces by using entourages instead of distances—could provide powerful tools for broader compactness and convergence contexts. This would align with general topological fixed-point principles like those in Kakutani–Glicksberg–Fan type theorems

2. **Probabilistic Metric Spaces**

In **probabilistic metric spaces**, distances are replaced by distribution functions, giving rise to fuzzy or uncertainty-driven frameworks Investigating whether our generalized contractive conditions and orbit-based completeness ideas can be adapted to such spaces could yield novel fixed-point results.

3. **Fuzzy and Fuzzy-Metric Spaces**

The rich literature on **fuzzy metric spaces** has applications in systems with vagueness and uncertainty, such as control theory and differential equations. Recent advances—like k-fuzzy frameworks and fuzzy 2-metric spaces—are particularly promising

4. **Multi-Valued and Ordered Structures**

Further developing fixed-point results for **multi-valued mappings** (beyond Nadler's extensions) and **partially ordered metric spaces** would enhance the theoretical reach, especially for applications in optimization, economics, and iterative algorithm convergence.

5. **Dynamic and Non-Deterministic Mappings**

Exploring fixed-point existence for **time-varying**, **randomized**, or **chaotic mappings** could be valuable in dynamical systems, computer science, and modeling complex adaptive systems.

Conclusion

The present study has significantly advanced the scope of fixed-point theory by formulating and proving two generalized theorems that extend beyond classical constraints. Traditional fixed-point results, such as the Banach contraction principle, rely heavily on assumptions of completeness and convexity in metric or normed spaces. While these conditions ensure mathematical elegance, they often fail to capture the complexity of real-world scenarios where such ideal structures rarely exist. Addressing this gap, the work introduces generalized fixed-point results that remain valid under weaker assumptions.

Theorem 1 establishes a fixed-point result for incomplete metric spaces under modified contraction conditions, thereby reducing the reliance on strict completeness. This contributes to the applicability of fixed-point techniques in diverse analytical and computational models, particularly those involving iterative approximations in spaces that are inherently non-complete. **Theorem 2**, on the other hand, extends the theory to non-convex sets, which play a central role in optimization and variational problems where convexity is often an unrealistic assumption. The proofs presented herein are constructed to maintain rigor while enhancing the theoretical flexibility of the framework.

Beyond the immediate results, the theorems offer a platform for multiple extensions. These include generalized contractions on partially ordered sets, fixed-point results for multi-valued mappings, and potential adaptations for probabilistic metric spaces and fuzzy systems. Such extensions hold immense potential in applied mathematics, particularly in control theory, computational algorithms, and economic modeling, where uncertainty and irregular structures dominate.

this work demonstrates that fixed-point theory is not confined to traditional rigid structures but can evolve to accommodate generalized, real-world conditions. The results obtained here not only broaden the mathematical foundations of the theory but also strengthen its relevance across scientific and engineering domains. Future research may continue to relax existing conditions and explore hybrid frameworks, ensuring that fixed-point methods remain at the forefront of both theoretical innovation and practical application.

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