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Calculation of Definite Integrals using Residues

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Abstract

This article presents the fundamental concepts of residue theory and demonstrates their application in the calculation of definite integrals, including certain types of improper integrals.

Keywords: singular point, isolated singularity, removable singularity, pole, residue of a function, definite integral, improper integral, closed contour, Jordan lemma.

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INTRODUCTION.

Solving many problems in physics and mathematics involves the calculation of integrals. In most cases, the fundamental theorems of residues are used to evaluate various types of integral expressions. In practice, Cauchy's theorem is convenient for calculating integrals taken over closed contours, while Jordan's lemma is useful for evaluating improper integrals.

Object of research and methods used. Suppose that the function $f(z)$ is holomorphic in $\{0 < |z - a| < d\}$, and the point a is an isolated special point of this function.

Definition 1 [1–4, 7]. This

$$\frac{1}{2pi}\oint_{|z-a|=r} f(z)dz \qquad (0 < r < d)$$

The integral of the function $f(z)$ at the point a is called the residue of $f(z)$ at and a is denoted by $\operatorname{res}_{z=a} f(z)$

$$\operatorname{res}_{z=a} f(z) = \frac{1}{2pi}\oint_{|z-a|=r} f(z)dz.$$

it is clear that if the function $f(z)$ is holomorphic at the point a, then $\operatorname{res}_{z=a} f(z) = 0$. Let the function

$f(z)$ be holomorphic in the domain $\{r < |z| < \bullet\}$.

Definition 2 [1–4, 7]. The integral

$$-\frac{1}{2\pi i}\oint_{|z|=r} f(z)dz \quad (r > r)$$

is called the residue of the function $f(z)$ at the point $z = \bullet$ and is denoted by $\operatorname{res}_{z=\bullet} f(z)$.

$$\operatorname*{res}_{z=\bullet} f(z) = -\frac{1}{2\pi i}\oint_{|z|=r} f(z)dz.$$

Now we present the formulas used to calculate the residues of a function [1–4].

1) If $z = a$ is a first-order pole of the function $f(z)$ at the point, then

$$\operatorname*{res}_{z=a} f(z) = \lim_{z\to a}(z-a)f(z) \qquad (1)$$

2) If the functions $f(z)$ and $y(z)$ for $f(z) = \frac{f(z)}{y(z)}$ are holomorphic at the point a, and $y(a) = 0, y'(a) \neq 0$, then

$$\operatorname*{res}_{z=a} f(z) = \frac{f(a)}{y'(a)} \qquad (2)$$

3) If point $z = a$ is an n -th pole of function $f(z)$, then

$$\operatorname*{res}_{z=a} f(z) = \frac{1}{(n-1)!}\lim_{z\to a}\frac{d^{n-1}\left((z-a)^n f(z)\right)}{dz^{n-1}} \qquad (3)$$

4) If the function $f(z)$ is holomorphic at a point $z = \bullet$, then

$$\operatorname*{res}_{z=\bullet} f(z) = \lim_{z\to\bullet} z\left(f(\bullet) - f(z)\right) \qquad (4)$$

5) If $f(z) = f\left(\frac{1}{z}\right)$ and the function $f(z)$ is holomorphic at the point $z = 0$, then

$$\operatorname*{res}_{z=\bullet} f(z) = -f'(0) \qquad (5)$$

1. Calculating integrals using deductions. Various integrals can be calculated using deductions. The following theorem plays an important role in this.

Theorem (Cauchy's theorem) [1-4]. Suppose,

1) The function $f(z)$ is holomorphic on the field $D \setminus \{a_1, a_2, ..., a_n\} \left(D \subset C, \quad a_1, a_2, ...a_n \in D\right)$,

2) the function $f(z)$ is defined up to the boundary of the domain and continuous in $\overline{D} \setminus \{a_1, a_2, ..., a_n\}$,

3) Let ∂D - be a closed contour that can be corrected. Then

$$\int_{\partial D} f(z)dz = 2\pi i \sum_{k=1}^{n} res_{z=a_k} f(z) \quad (6)$$

is valid.

Note. Formula (6) is also valid for the case $\infty \in D$. Only in this case it is sufficient to consider $z = \infty$ as a special point for $f(z)$ and to take the orientation of the line ∂D - in the clockwise direction.

The above Cauchy theorem is used in practice to calculate integrals taken over a closed contour.

2. Calculating definite integrals using deductions.

Definite integrals can also be calculated using deductions. In this case, the definite integral is calculated by reducing it to the integral taken along the contour of a function of a complex variable

a) Calculate integrals of the form $\int_0^{2\pi} R(\cos x, \sin x)dx$ **.**

Given this

$$I = \int_0^{2\pi} R(\cos x, \sin x)dx \quad (7)$$

integral, it is required to calculate it, where $R(\cos x, \sin x)$ is a rational function of - $\cos x$ and $\sin x$ and is continuous on [0,2]. Considering that $\cos x = \frac{e^{ix} + e^{-x}}{2}, \quad \sin x = \frac{e^{ix} - e^{-x}}{2i}$ according to Euler's formula, and then introducing the notation $z = e^{ix}$

then

$$x \in [0, 2\pi] \Rightarrow z \in \{z \in C : |z| = 1\}, \cos x = \frac{1}{2}\left(z + \frac{1}{z}\right), \quad \sin x = \frac{1}{2i}\left(z - \frac{1}{z}\right), \quad dx = \frac{1}{iz}dz$$

and the given (7)-integral becomes $I = \int_0^{2\pi} R(\cos x, \sin x)dx = \int_{|z|=1} R_1(z)dz$ as follows where $R_1(z) = \frac{1}{iz} R\left(\frac{1}{2}\left(z + \frac{1}{z}\right), \frac{1}{2i}\left(z - \frac{1}{z}\right)\right)$. The resulting integral is calculated using formula (6).

b) Calculating improper integrals. Improper integrals can also be calculated using the theory of deductions. This is based on the following theorem.

Theorem [1-4]. Let the function $f(z)$ be holomorphic at all points of the domain $\{z \in C : \mathrm{Im} z > 0\}$

except for a finite number of special points and continuous on its boundary.

If $\lim_{r\to\infty}\int_{g_r} f(z)dz = 0 \quad \left(g_r = \{|z| = r, 0 \le \arg z \le p\}\right)$ (8)

then $\int_{-\infty}^{+\infty} f(x)dx$ is an approximation and

$$\int_{-\infty}^{+\infty} f(x)dx = 2pi \sum_{\mathrm{Im}\, z>0} \operatorname{res}_{z=z_k} f(z). \quad (9)$$

The following lemmas are used to show that condition (8) in this theorem holds.

Lemma 1 (Jordan's Lemma) [1-4]. If

$$\lim_{r\to\infty} \max_{z\in g_r} |f(z)| = 0 \quad (10)$$

then

$$\lim_{r\to\infty} \int_{g_r} f(z)dz = 0 \quad (11)$$

Lemma 2 (Jordan's Lemma) [1-4]. If

$$\lim_{r\to\infty} \max_{z\in g_r} |f(z)| = 0 \quad (12)$$

then for $\forall \lambda > 0 \lim_{r\to\infty} \int f(\mathrm{z})\, \mathrm{e}^{i\lambda z}\, dz = 0$ (13)

Now let's look at improper integrals of the form $\int_{-\infty}^{+\infty} e^{i\lambda x} R(\mathrm{x})dx$. If $\lim_{r\to\infty} \max_{z\to\gamma} |R(\mathrm{z})| = 0$, then applying Jordan's 2nd lemma and the above theorem to this integral, we obtain the following formulas:

$$\int_{-\infty}^{+\infty} R(\mathrm{x})\cos\lambda x dx = -2\pi \cdot \mathrm{Im}\left\{ \sum_{\mathrm{Im} z_k \,\mathrm{f}\, 0} \operatorname{res}_{z=z_k} \left[e^{i\lambda z} \cdot R(\mathrm{z}) \right] \right\} \quad (14)$$

$$\int_{-\infty}^{+\infty} R(\mathrm{x})\sin\lambda x dx = 2\pi \cdot \mathrm{Re}\left\{ \sum_{\mathrm{Im} z_k \,\mathrm{f}\, 0} \operatorname{res}_{z=z_k} \left[e^{i\lambda z} \cdot R(\mathrm{z}) \right] \right\} \quad (15)$$

RESULTS AND THEIR ANALYSIS

Now we will consider deductions and the problems of calculating integrals using them [5-8].

1. Calculate the discount of the function $f(\mathrm{z}) = \dfrac{\cos 2z}{(\mathrm{z}-1)^3}$. Since the special point $z = 1$ of the function is a 3rd-order pole point, according to formula (3), we obtain:

$$\operatorname{res} f(\mathrm{z}_0) = \frac{1}{2!}\lim_{z\to 1}\frac{d^2}{dz^2}\left(\frac{\cos 2z}{(z-1)^3}(z-1)^3\right) = \frac{1}{2}\lim_{z\to 1}(\cos 2z)'' = \frac{1}{2}\lim_{z\to 1}(-2\sin 2z)'$$

$$= \frac{1}{2}\lim_{z\to 1}(-4\cos 2z) = -2\cos 2.$$

Calculate the following integral using deductions.

The function $\oint_{|z|=4} \frac{e^z-1}{z^2+z}dz$ under the integral is analytic in all parts of the domain $|z|<4$ except for the points $z_0=0$ and $z_1=-1$. Then, according to the deduction theorem, we write:

$$\oint_{|z|=4} \frac{e^z-1}{z^2+z}dz = 2\pi i \cdot (resf(0)+resf(-1)).$$

$$\lim_{z\to 0}\frac{e^z-1}{z(z+1)} = \lim_{z\to 0}\frac{1+z+\frac{z^2}{2!}+\frac{z^3}{3!}+...-1}{z(z+1)} = \lim_{z\to 0}\frac{z\left(1+\frac{z}{2!}+\frac{z^2}{3!}+...\right)}{z(z+1)} = 1$$

the point $z_0=0$ is a special point to be $resf(0)=0$ eliminated. Therefore, since the point is a $z_1=-1$ 1st-order pole, we calculate the deduction using formula (2):

$$resf(-1) = \left.\frac{e^z-1}{(z^2+z)'}\right|_{z=-1} = \left.\frac{e^z-1}{2z+1}\right|_{z=-1} = 1-e^{-1}$$

Therefore, $\oint_{|z|=4} \frac{e^z-1}{z^2+z}dz = 2\pi i(1-e^{-1})$. Calculate the following definite integral $\int_0^{\pi}\frac{\cos^2 x}{2-\sin^2 x}dx$ using If we make the substitution $e^{2ix}=z$ in this integral, we get

$$x\in[o,\pi]\Rightarrow z\in\{z\in C:|z|=1\},$$

$$dx=\frac{1}{2iz}dz,$$

$$\cos^2 x=\frac{1+\cos 2x}{2}=\frac{1+\frac{1}{2}(z+\frac{1}{z})}{2}$$

$$\sin^2 x=\frac{1-\cos 2x}{2}=\frac{1-\frac{1}{2}(z+\frac{1}{z})}{2}$$

and the equality

$$\int_0^{\pi}\frac{\cos^2 dx}{2-\sin^2 x}=\frac{1}{2i}\oint_{|z|=1}\frac{1}{z}\cdot\frac{\frac{1+\frac{1}{2}(z+\frac{1}{z})}{2}}{2-\frac{1-\frac{1}{2}(z+\frac{1}{z})}{2}}dz=\frac{1}{2i}\oint_{|z|=1}\frac{1}{z}\cdot\frac{(z+1)^2}{z^2+6z+1}dz$$

holds. Calculate the following integral with an infinite limit $\int_0^{+\infty}\frac{dx}{\left(x^2+1\right)^n}$ $\left(n\in N\right)$ using deductions.

First, we write the given integral in the form $\int_0^{+\infty}\frac{dx}{\left(x^2+1\right)^n}=\frac{1}{2}\int_{-\infty}^{+\infty}\frac{dx}{\left(x^2+1\right)^n}$. Now, if we say

$f(z)=\frac{1}{\left(z^2+1\right)^n}=\frac{1}{(z+i)^n(z-i)^n}$, this function has a special point $z=i$ at $\{z\in C:\operatorname{Im}z>0\}$, and an n-order pole.Clearly,

$$\lim_{r\to\infty} r\cdot\max_{\gamma_r} f(z)=0 \qquad \{|z|=r,\ 0\le\arg z\le\pi\}(\gamma_r=)\Rightarrow$$

According to Jordan's lemma 1, $\lim\limits_{r\to\infty}\int\limits_{\gamma_r} f(z)dz=0$. Then, according to the theorem in point 2, $\int\limits_{-\infty}^{+\infty}\frac{dx}{\left(x^2+1\right)^n}=2\pi i\operatorname*{res}_{z=i} f(z)$. Using formula (3), we find:

$$\operatorname*{res}_{z=i} f(z)=\lim_{z\to i}\frac{1}{(n-1)!}\frac{d^{n-1}}{dz^{n-1}}\left[(z-i)^n\cdot f(z)\right]=$$

$$=\frac{1}{(n-1)!}\lim_{z\to i}\frac{d^{n-1}}{dz^{n-1}}\left[\frac{1}{(z+i)^n}\right]=\frac{(2n-3)!!}{(2n-2)!!}\cdot\frac{1}{2i}.$$

As a result, $\int\limits_{-\infty}^{+\infty}\frac{dx}{\left(x^2+1\right)^n}=2\pi i\cdot\frac{1}{2i}\frac{(2n-3)!!}{(2n-2)!!}=\frac{(2n-3)!!}{(2n-2)!!}\cdot\pi$ and we find that for the given integral,

$$\int\limits_{0}^{+\infty}\frac{dx}{\left(x^2+1\right)^n}=\frac{(2n-3)!!}{(2n-2)!!}\cdot\frac{\pi}{2}.$$

CONCLUSION. From the above, it can be concluded that the theory of deductions is of great importance in the calculation of integrals, and its use in teaching the subject of integrals as much as possible creates a number of conveniences and gives effective results in mastering the subject.

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